International Journal of Mathematics, Statistics and Operations Research
Vol. 5; Number 2; 2025, pp. 209-222
https://doi.org/10.47509/IJMSOR.2025.v05i02.02

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www.arfjournals.com

# Uniform-in-time bounds of Flocculation System Type with Diffusion

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[RO]*L. Chergui, S. Zermani* [LO]*Journal of Econometrics and Statistics*

**ABSTRACT**
This paper deals with a diffusion-advection system modeling flocculation process in a un-stirred chemostat. We prove that there is a finite upper bound on the total bacteria density which enables us to prove existence of global solution. Then, we show existence of non-trivial positive solutions to the corresponding steady-state system. Also, the uniform stability of the constant solution is investigated.

**KEYWORDS**
flocculation, global solution, lower-upper solutions, microbial growth, spectral theory, steady solution

## 1. Introduction

This paper is devoted to the study of the following Reaction-Diffusion system

$$\begin{cases} S_t = d_0 S_{xx} - S_x - f(S)u - g(S)v, \quad \text{on } (0,1)\times(0,T), \\ u_t = d_1 u_{xx} - u_x + f(S)u - \dfrac{1}{y_u}\alpha(u,v)u + \beta(u,v)v, \quad \text{on } (0,1)\times(0,T), \\ v_t = d_2 v_{xx} - v_x + g(S)v + \alpha(u,v)u - \dfrac{1}{y_v}\beta(u,v)v,, \quad \text{on } (0,1)\times(0,T), \end{cases} \tag{1}$$

with boundary conditions:

$$\begin{cases} -d_0 S_x(0,t) + S(0,t) = \gamma_S, \ S_x(1,t) = 0, \\ -d_1 u_x(0,t) + u(0,t) = \gamma_u, \ u_x(1,t) = 0, \\ -d_2 v_x(0,t) + v(0,t) = \gamma_v, \ v_x(1,t) = 0, \end{cases} \tag{2}$$

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**Article History**
Received : 21 June 2025; Revised : 27 July 2025; Accepted : 01 August 2025; Published : 10 November 2025

**To cite this paper**

Lassaad Cherguia (2025). Uniform-in-time bounds of Flocculation System Type with Diffusion. *International Journal of Mathematics, Statistics and Operations Research.* 5(2), 209-222.

and initial conditions:

$$S(x,0) = w_S(x),\ u(x,0) = w_u(x),\ v(x,0) = w_v(x),\ x \in [0,1]. \tag{3}$$

In our study, we focus on dynamics in the region $S \geq 0,\ u \geq 0$ and $v \geq 0$ corresponding to biologically meaningful solutions.

So, here $d_0, d_1, d_2, y_u, y_v > 0,\ \gamma_S, \gamma_u, \gamma_v \geq 0,\ w := \begin{pmatrix} w_S \\ w_u \\ w_v \end{pmatrix}$ is a component-wise bounded and nonnegative function on $(0,1)$, and $f, g : \mathbb{R}_+ \to \mathbb{R}_+$ and $\alpha, \beta : \mathbb{R}_+^2 \to \mathbb{R}_+$ are locally Lipschitz.

Let us recall that existence of a non-negative solutions of (1)-(3) is proved in [14] under sufficient conditions for a specific class of flocculation and deflocculation rates. In this paper, we extend the class of $\alpha(\cdot), \beta(\cdot)$ for more range of model's parameters to improve the results of [4, 14]. It turns out that we can obtain global existence of componentwise nonnegative solutions to (1)-(3) under the condition $y_u y_v \leq 1$ provided $\alpha(\cdot), \beta(\cdot)$ are nonnegative, and either $\alpha(u,v) \leq K(u+v+1)$ and $\beta(u,v)$ is polynomially bounded above, or $\beta(u,v) \leq K(u+v+1)$ and $\alpha(u,v)$ is polynomially bounded above. The results have no dependence on the degree of the polynomial bound, and actually, the linear bounds can be super linear.
Also, by developing a monotone method, we improve the results of [4, 14], where the flocculation rates in [4, 14] are bounded and the per-capita growth are monotone. Indeed, inspired by [10, 13], the existence of a non-trivial solution will be established by means of the monotone method based on constructing a lower and an upper solution for the problem (13).
So, we assume that flocculation-deflocculation rates satisfy

($\mathbf{H_1}$) $y_u y_v < 1$ and there exists a constant $a \geq 1$ and $K_1 > 0$ satisfying:

$$\alpha(u,v)u + \beta(u,v)v \geq K_1(u+v)^a,\ \text{for all } u, v \geq 0.$$

($\mathbf{H_2}$) There exists a constant $K > 0$ and $1 \leq r, s < 3$ satisfying:

$$\begin{cases} -\dfrac{1}{y_u}\alpha(u,v)u + \beta(u,v)v \leq K(u+v+1)^r,\ \text{for all } u, v \geq 0, \\ \alpha(u,v)u - \dfrac{1}{y_v}\beta(u,v)v \leq K(u+v+1)^s,\ \text{for all } u, v \geq 0. \end{cases}$$

Here $S(t)$ is the substrate concentration, $u(t)$ and $v(t)$ denote respectively the concentrations of isolated and attached bacteria at time $t$, $f(\cdot)$ and $g(\cdot)$ represent, respectively, the per-capita growth rates of the isolated and attached bacteria, $\alpha(\cdot)$ and $\beta(\cdot)$ denote, respectively, the flocculation and deflocculation rates. This system is modeling flocculation process in a un-stirred chemostat where the isolated or planktonic bacteria naturally aggregate, reversibly, to one another to form macroscopic flocs. These processes were and still are subject of a lot of both theoretical and experimental researches, [2]-[6], [9]-[15]. The coefficients $y_u$ and $y_v$ are positive constants that respectively consider the characteristics of the medium, the efficiency of collision and the yield coefficient for free and attached bacteria. The isolated and attached microbial cells are assumed to be capable of random movement, modeled by diffusion with diffusivity constant $d$, see [14].

A key feature of the model first formulated by Freter [7, 8] for the chemostat. In [12], B. Tang et all considered the cases $\alpha(\cdot) = \alpha(S),\ \beta(\cdot) = \beta(S)$, while, R. Freter et all, in [8], studied the phenomena for

$$\alpha(\cdot) = a(1-W),\ \beta(\cdot) = b + g(S)(1 - G(W)),\ \text{with, } W = \frac{v}{v_{max}}, G(W) = \frac{1-W}{k+1-W}.$$

And recently, the authors in [5]-[6], are interested in the study of chemostat model where $\alpha(\cdot) = a(u+v),\ \beta(\cdot) = b$ but they considered a model that is only driving by reaction systems representing instantaneous interactions in which spatial effects are considered negligible due to concentrations homogeneously distributed, described as follows,

$$\begin{cases} S_t = D(S_{in} - S) - f(S)u - g(S)v \\ u_t = (f(S) - D_0)u - \alpha(\cdot)u + \beta(\cdot)v \\ v_t = (g(S) - D_1)v + \alpha(\cdot)u - \beta(\cdot)v. \end{cases}$$

The manuscript is organized as follows. Section two summarizes the main results established in this paper. Section three presents the tools needed for the existence results proved here. Section four is devoted to proving well-posedness of the problem (1)-(3) and the associated steady state in the case of $d_0 = d_1 = d_2$.

## 2. Main results

First, under appropriate conditions we prove that the system (1)-(3) has a classical global solution.

**Theorem 1.** *Suppose* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then for any non-negative, bounded initial data* $w$*, there exists a positive global solution* $(S, u, v)$ *to (1)-(3) satisfying*

$$\sup_{t\geq 0} \|u + v\|_{L^\infty(0,1)} < +\infty.$$

In order to investigate the asymptotic behavior of solution of (1)-(3), we consider the eigenvalue problem introduced in [3]:

$$(P_\lambda)\begin{cases} \lambda\varphi = d\varphi'' - \varphi'. \\ -d\varphi'(0) + \varphi(0) = 0,\ \varphi'(1) = 0, \end{cases}$$

where $d$ is a positive constant. The eigenvalues $\{\lambda_n\}_{n\geq 0}$ of $(P_\lambda)$ satisfy $\lambda_{n+1} < \lambda_n,\ \forall n \geq 0$ and $\lambda_0 < -1$. In order to emphasize the dependance of $\lambda_0$ on $d$ and take account of its sign, let us denote $\lambda_d = -\lambda_0$. Then, using the following assumption:

$(\mathbf{H_3})$ There exists a positive constant $k \geq 1$ such that

$$\frac{1}{k}\beta(u,v) \leq \alpha(u,v) \leq k\beta(u,v), \quad \text{for all, } u, v \geq 0, (\text{ that is } \alpha(u,v) \approx \beta(u,v)).$$

and

$$\limsup_{x\to+\infty} \alpha(x,x)(k - \min(\frac{1}{y_u}, \frac{1}{y_v})) \le \lambda_d, \tag{4}$$

we show the existence of a steady-state solution with microorganisms present in the medium, that is with $u > 0$ and $v > 0$.

**Theorem 2.** *Assume* $(\mathbf{H_3})$ *such that* $f(1) > \lambda_d$ *and* $g(1) > \lambda_d$*. Then, there exists a non-trivial steady-state solution* $(S, u, v)$ *of* $(1) - (3)$*, that is* $(S, u, v) \neq (\gamma_S, 0, 0)$*.*

Finally, we investigate in the asymptotic stability of the constant solution $E_0 = (\gamma_S, 0, 0)$, under some assumptions on the growth rates $f$ and $g$ and the diffusion coefficient $d$.

**Theorem 3.** *Assume* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$*. Then, if the first eigenvalue* $-\lambda_d$ *satisfies*

$$\lambda_d > \max(f(1), g(1)),$$

*the washout steady-state* $E_0 = (\gamma_S, 0, 0)$ *is uniformly asymptotically stable.*
*If*

$$\lambda_d < \max(f(1), g(1)),$$

*then, the washout steady-state is unstable.*

## 3. Tools

### 3.1. *Global existence*

Our first result states that there is a finite upper bound on the total isolated bacteria density $\int_0^1 u(x,t)dx$ and on the total attached bacteria density $\int_0^1 v(x,t)dx$ at time $t$, independently of diffusion coefficients $d_0$, $d_1$ and $d_2$.

**Lemma 1.** *Suppose* $(\mathbf{H_1})$ *holds. Then, there exists* $L > 0$ *such that for all* $t \geq 0$*, we have*

$$\int_0^1 (u(x,t) + v(x,t))dx \le L.$$

***Proof.*** Set $b = max\{1 + \frac{1}{y_u}, 1 + \frac{1}{y_v}\}$, multiply the $S$ equation by $b$, the $u$ equation by $1 + \frac{1}{y_v}$ and the $v$ equation by $1 + \frac{1}{y_u}$. Then, integrate these equations over $(0, 1)$, sum them, so we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^1 bS+(1+\frac{1}{y_v})u+(1+\frac{1}{y_y})vdx &\leq b\gamma_S+(1+\frac{1}{y_v})\gamma_u+(1+\frac{1}{y_u})\gamma_v \\ &+ \int_0^1(1-\frac{1}{y_uy_v})[\alpha(u,v)u+\beta(u,v)v]dx \\ &\leq \gamma+\int_0^1(1-\frac{1}{y_uy_v})[\alpha(u,v)u+\beta(u,v)v]dx,\end{aligned}$$

where $\gamma := b\gamma_S+(1+\frac{1}{y_v})\gamma_u+(1+\frac{1}{y_u})\gamma_v$. Then, thanks to $(\mathbf{H_1})$, we have

$$\frac{d}{dt}\int_0^1 bS+(1+\frac{1}{y_v})u+(1+\frac{1}{y_y})vdx \leq \gamma+K_1(1-\frac{1}{y_uy_v})(\int_0^1[u+v])^a dx.$$

Let us define $z(t)=\int_0^1[bS+(1+\frac{1}{y_v})u+(1+\frac{1}{y_y})v]dx$, then we have

$$z'(t)\leq\gamma+K_1(1-\frac{1}{y_uy_v})(\int_0^1\frac{(1+\frac{1}{y_v})u+(1+\frac{1}{y_u})v}{2+\frac{1}{y_u}+\frac{1}{y_v}}dx)^a.$$

Set $\kappa := K_1\frac{\frac{1}{y_uy_v}-1}{(2+\frac{1}{y_u+}\frac{1}{y_v})^a}$, then we have

$$z'\leq\gamma-\kappa(z(t)-(\int_0^1 bSdx)^a.$$

Since $S$ is uniformly bounded, $\limsup_{t\to+\infty} S(t,x)\leq\gamma_S$, then there exists a positive constant $M$ so that $\int_0^1 bSdx\leq M$. suppose that there exists $t_1$ such that $z(t_1)>M+1$, then there exists $t_0$ such that $t_0=0$ or $z(t_0)=M+1$ and for all $t\in I=]t_0,t_1[,\ z(t)\geq M+1$, thus

$$z'(t)\leq\gamma-\kappa(z(t)-M)^a.$$

Now, let us define $\tilde{z}(t):=z(t)-M$, then since $a\geq 1$, we have for all $t\in I=]t_0,t_1[$,

$$\tilde{z}'(t)\leq\gamma-\kappa(\tilde{z}(t))^a\leq\gamma-\kappa\tilde{z}(t),$$

then, we obtain that for all $t\in I=]t_0,t_1[$, $z(t)\leq z(t_0)+\frac{\gamma}{\kappa}$, due to $\kappa>0$.
This implies that for all $t\geq 0$, $z(t)$ is uniformly bounded, which ends the proof. □

In what follows, we assume that there is no input of microorganisms from the inflow, that is, $\gamma_u=\gamma_v=0$, which allow us to obtain the case of the washout steady state $(\gamma_S,0,0)$. Furthermore, we prove that there is a finite upper bound on the total energy.

**Lemma 2.** *Suppose* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold and let* $p \geq 2$. *Then, there exists a positive constant* $C_p$ *such that,*

$$\int_0^1 (u^p(x,t) + v^p(x,t))dxdt \leq C_p, \text{ for all } \tau \geq 0.$$

***Proof.*** Let $u$ and $v$ solve $(1)-(3)$ and let us define the $L^p-$energy functions by

$$\mathcal{L}_p(t) = \int_0^1 (u^p(x,t) + v^p(x,t))dx.$$

So, we have

$$\begin{aligned}
\mathcal{L}'_p(t) &= p\int_0^1 (u_t u^{p-1} + v_t v^{p-1})dx & (5)\\
&= p\int_0^1 u^{p-1}(d_1 u_{xx} - u_x & (6)\\
&+ f(S)u - \frac{1}{y_u}\alpha(u,v)u + \beta(u,v)v) & (7)\\
&+ p\int_0^1 v^{p-1}(d_2 v_{xx} - v_x + g(S)v & (8)\\
&+ \alpha(u,v)u - \frac{1}{y_v}\beta(u,v)v)dx. & (9)
\end{aligned}$$

We can assume that $r \geq s$, so by integration by parts and thanks to the polynomial growth $(\mathbf{H_2})$, we get

$$\begin{aligned}
\mathcal{L}'_p(t) &\leq -p(p-1)(d_1\int_0^1 u^{p-2}u_x^2 dx + d_2\int_0^1 v^{p-2}v_x^2 dx)\\
&- u^p(1,t) - (p-1)u^p(0,t) - v^p(1,t) - (p-1)v^p(0,t) + p\int_0^1 f(S)u^p dx\\
&+ p\int_0^1 g(S)v^p dx + pK\int_0^1 u^{p-1}(1+u+v)^r dx + pK\int_0^1 v^{p-1}(1+u+v)^r dx\\
&\leq -p(p-1)(\int_0^1 [d_1 u^{p-2}u_x^2 + d_2 v^{p-2}v_x^2]dx)\\
&+ p\int_0^1 [f(S)u^p + g(S)v^p]dx + pK\int_0^1 (1+u+v)^{p-1+r}dx
\end{aligned}$$

Let $d := \min(d_1, d_2)$, $\alpha_p = dp(p-1)$ and $M_f = \max\limits_{0\leq S\leq\gamma_S}(f(S))$, $M_g = \max\limits_{0\leq S\leq\gamma_S}(g(S))$. Using that for all $r \geq 1$, we have that

$$u^p \leq 1 + u^{p-1+r},$$

then there exist a positive constant $C$ that depends on $p, M_f, M_g, r$ and $K$, such that

$$\mathcal{L}'_p(t) + \frac{\alpha_p}{2}\int_0^1 (u^{p-2}u_x^2 + v^{p-2}v_x^2)dx \leq C(1 + \int_0^1 u^{p-1+r} + v^{p-1+r}dx).$$

Adding $\alpha_p \int_0^1 (u^p + v^p)dx$ to the both sides above, we get for $C$ large enough that

$$\mathcal{L}_p'(t) + \frac{\alpha_p}{2}\int_0^1 u^{p-2}u_x^2 + v^{p-2}v_x^2 + \alpha_p \int_0^1 u^p + v^p dx) \leq C(1 + \int_0^1 u^{p-1+r} + v^{p-1+r} dx).$$

Then, thanks to $L^1$-estimates, we can apply the Gagliardo-Nirenberg inequality, then there exist $C_p > 0$ and $0 < \alpha < 1$ satisfying

$$\int_0^1 (u(x,t))^p \, dx \leq C_p(\int_0^1 \left|\left(u(x,t)^{p/2}\right)_x\right|^2 dx)^{\alpha}(\int_0^1 |u(x,t)|^{\frac{p}{2}} \, dx)^{2(1-\alpha)}.$$

If $\|u(.,t)\|_{p/2,(0,1)} \leq M$, from Young's inequality, for $\varepsilon > 0$ there exists $C_{p,\varepsilon,M} > 0$, independent of $u$, so that

$$\int_0^1 |u(x,t)|^p \, dx \leq \varepsilon \int_0^1 \left|\left(u(x,t)^{p/2}\right)_x\right|^2 dx + C_{p,\varepsilon,M}. \tag{10}$$

Since $1 \leq r < 3$, then there exists $C_p > 0$ depending on $p, r, C$ such that

$$C\int_0^1 (u^{p-1+r} + v^{p-1+r})dx \leq \frac{\alpha_p}{2}(\int_0^1 u^{p-2}u_x^2 + v^{p-2}v_x^2 + \int_0^1 u^p + v^p dx) + C_p.$$

Thus, we obtain that

$$\mathcal{L}_p'(t) + \frac{\alpha_p}{2}\mathcal{L}_p(t) \leq C_p,$$

and this implies, since $\alpha_p > 0$, the uniform-in-time boundedness of $\mathcal{L}_p$.
That is, $\sup_{t\geq 0} \mathcal{L}_p \leq C_p$, which ends the proof.

□

### 3.2. *Steady state*

In this section, we take $\gamma_S = 1$, $\gamma_u = \gamma_v = 0$ and $d_0 = d_1 = d_2$. First, let us remark that the solution $E_0 = (S, u, v) = (1, 0, 0)$, called the washout steady-state, is the unique constant steady-state solution of

$$\begin{cases} dS_{xx} - S_x = f(S)u + g(S)v \\ du_{xx} - u_x = -f(S)u + \frac{1}{y_u}\alpha(u,v)u - \beta(u,v)v \\ dv_{xx} - v_x = -g(S)v - \alpha(u,v)u + \frac{1}{y_v}\beta(u,v)v \end{cases} \tag{11}$$

with boundary conditions

$$\begin{cases} -dS_x(0) + S(0) = 1, \\ -du_x(0) + u(0) = -dv_x(0) + v(0) = 0, \\ S_x(1) = 0, \; u_x(1) = 0, \; v_x(1) = 0. \end{cases} \tag{12}$$

Then, we rewrite the system (11) – (12) in the vectorial form

$$\begin{cases} Dw_{xx} - w_x + F(w) = 0, \ x \in (0,1), \\ -dw_x(0) + w(0) = 0, \ w_x(1) = 0. \end{cases} \tag{13}$$

where

$$w = (1-S, u, v)^T, \ D := \begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{pmatrix}$$

and

$$F(w) := \begin{pmatrix} f(S)u + g(S)v \\ f(S)u - \frac{1}{y_u}\alpha(u,v)u + \beta(u,v)v \\ g(S)v + \alpha(u,v)u - \frac{1}{y_v}\beta(u,v)v \end{pmatrix}.$$

We say that $w \in C^1([0,1])$ is a lower solution for the problem (13) if $w$ satisfies

$$\begin{cases} Dw_{xx} - w_x + F(w) \geq 0, \ x \in (0,1), \\ -dw_x(0) + w(0) \leq 0, \ w_x(1) = 0. \end{cases}$$

The vectorial inequality $U \leq V$ means that the component $U_i \leq V_i$, for each $i \in \{1,2,3\}$. If the inequality above is reversed, $w$ is called an upper solution.
In the sequel, we suppose that there exist a lower solution $\underline{w} = (\underline{S}, \underline{u}, \underline{v})$ and an upper solution $\overline{w} = (\overline{S}, \overline{u}, \overline{v})$ for the problem (13), such that $\underline{S} \leq \overline{S}, \underline{u} \leq \overline{u}, \underline{v} \leq \overline{v}$ on $[0,1]$ and to develop a monotone method for the problem (13), we assume that there exists a real $\lambda > 0$ such that

$$\lambda x_i + F_i(.,x_i,.) \leq \lambda y_i + F_i(.,y_i,.), \ i \in \{1,2,3\}, \ \text{for } \underline{w}_i \leq x_i \leq y_i \leq \overline{w}_i.$$

Let us define the operator $T$ on $(C^1([0,1]))^3$ by $Tw = z$, where $z$ is the unique solution of the following problem

$$\begin{cases} Dz_{xx} - z_x - \lambda z = -\lambda w - F(w), \ x \in (0,1), \\ -dz_x(0) + z(0) = 0, \ z_x(1) = 0. \end{cases}$$

That is $z(x) = Tw(x) = \int_0^1 G(x,t)[\lambda w(t) + F(w(t))]dt$, We note that the Green's functions $G$, is positive (see e.g. Theorem 4.2 of [1]). Furtheremore, $T$ is a nondecreasing operator on $[\underline{w}, \overline{w}]$ and that $w$ is a solution of (13) if and only if $Tw = w$.
In what follows, by a monotone method, we prove existence solutions to the problem (13).

**Lemma 3.** *If $U$ and $V$ are ,respectively, a lower and an upper solutions for the problem (13), in $[\underline{w}, \overline{w}]$, then $TU$ is a lower solution for the problem (13) satisfying $U \leq TU$ and $TV$ is an upper solution for the problem (13) satisfying $TV \leq V$.*

***Proof.*** Let $U$ be a lower solution for the problem (13) such that $\underline{w} \leq U \leq \overline{w}$. Put

$w = TU - U$, then we have

$$\begin{cases} Dw_{xx} - w_x - \lambda w = -F(U) - DU_{xx} + U_x \leq 0, \ x \in (0,1), \\ -dw_x(0) + w(0) = 0, \ w_x(1) = 0. \end{cases}$$

Which yields that, $w \geq 0$ on $[0,1]$. That is $U \leq TU$ on $[0,1]$ and similarly, we prove that $TV \leq V$ on $[0,1]$.
Now, since we have that $\underline{w} \leq U \leq TU \leq T\overline{w} \leq \overline{w}$, then $TU$ satisfies that

$$\begin{cases} D(TU)_{xx} - (TU)_x + F(TU) = [\lambda TU + F(TU)] - [\lambda U + F(U)] \geq 0, \ x \in (0,1), \\ -d(TU)_x(0) + TU(0) = 0, \ (TU)_x(1) = 0. \end{cases}$$

Which implies that $TU$ is a lower solution for the problem (13) and similarly, we prove that $TV$ is an upper solution for the problem (13). □

**Lemma 4.** *Let $\underline{w} = (\underline{S}, \underline{u}, \underline{v})$ and $\overline{w} = (\overline{S}, \overline{u}, \overline{v})$ be, respectively, a lower solution and an upper solution for the problem (13), such that $\underline{S} \leq \overline{S}, \underline{u} \leq \overline{u}, \underline{v} \leq \overline{v}$ on $[0,1]$. Then, there exist two extremal solutions in $[\underline{w}, \overline{w}]$ of the problem (13).*

***Proof.*** We define $(\underline{w}_n)_{n\in\mathbb{N}}$ and $(\overline{w}_n)_{n\in\mathbb{N}}$ as follows $\underline{w}_0 = \underline{w}$, $\overline{w}_0 = \overline{w}$,

$$\underline{w}_{n+1} = T\underline{w}_n \text{ and } \overline{w}_{n+1} = T\overline{w}_n.$$

Using the previous lemma, we obtain that

$$\underline{w} = \underline{w}_0 \leq .... \leq \underline{w}_n \leq \underline{w}_{n+1} \leq \overline{w}_{n+1} \leq \overline{w}_n \leq ... \leq \overline{w}_0 = \overline{w}.$$

Hence, the sequences $(\underline{w}_n)_{n\in\mathbb{N}}$ and $(\overline{w}_n)_{n\in\mathbb{N}}$ converge, respectively, to functions $W$ and $\widetilde{W}$ such that $\underline{w} \leq W \leq \widetilde{W} \leq \overline{w}$. It follows that

$$\lambda\underline{w} + F(\underline{w}) \leq \lambda W + F(W) \leq \lambda\widetilde{W} + F(\widetilde{W}) \leq \lambda\overline{w} + F(\overline{w}).$$

So, by dominated convergence theorem, we obtain that $W = TW$ and $\widetilde{W} = T\widetilde{W}$. Which ends the proof.
If $w$ is solution of the problem (13) such that $\underline{w} \leq w \leq \overline{w}$, then we have for every $n \in \mathbb{N}$ that $T^n\underline{w} \leq T^n w \leq T^n\overline{w}$, which implies that $\underline{w}_n \leq w \leq \overline{w}_n$. Thus, $W \leq w \leq \widetilde{W}$. That is $W$ and $\widetilde{W}$ are extremal solutions of the problem (13). □

It is convenient to make the change of variables $\widetilde{S} = 1 - S$ and we will always interpret $1 - \widetilde{S}$ as the positive part of it: $(1 - \widetilde{S})_+$. So that $\widetilde{S}$ satisfies homogeneous boundary conditions. We then have, from (11)-(12), the system,

$$\begin{cases} -d\widetilde{S}_{xx} + \widetilde{S}_x = f(1-\widetilde{S})u + g(1-\widetilde{S})v \\ -du_{xx} + u_x = f(1-\widetilde{S})u + \beta(u,v)v - \frac{1}{y_u}\alpha(u,v)u \\ -dv_{xx} + v_x = g(1-\widetilde{S})v + \alpha(u,v)u - \frac{1}{y_u}\beta(u,v)v \end{cases} \tag{14}$$

with boundary conditions

$$\begin{cases} -d\widetilde{S}_x(0) + \widetilde{S}(0) = 0, \\ -du_x(0) + u(0) = -dv_x(0,t) + v(0) = 0, \\ \widetilde{S}_x(1) = 0,\ u_x(1) = 0,\ v_x(1) = 0, \end{cases} \tag{15}$$

**Lemma 5.** *Assume* ($\mathbf{H_1}$). *Let* $K$ *be the positive cone in* $(C([0,1],\mathbb{R}^+))^3$ *and* $(\overline{S},u,v) \in K$ *satisfying (14)-(15). Then, we have* $0 \le \widetilde{S}(x) \le 1,\ x \in [0,1]$.

***Proof.*** To get the ultimate boundedness of $\widetilde{S}$, multiplying the first equation in (14) by $\exp(\frac{-x}{d})$ and integrating, using the boundary conditions, leads to

$$\widetilde{S}'(x) = d^{-1}\int_x^1 \exp(\frac{x-t}{d})[f(1-\widetilde{S})u + g(1-\widetilde{S})v]dt. \tag{16}$$

Then $\widetilde{S}$ must satisfy

$$\widetilde{S}(x) = \int_0^1 \exp(\frac{\min(x,t)-t}{d})[f(1-\overline{S})u + g(1-\widetilde{S})v]dt. \tag{17}$$

Let us remark that $\widetilde{S}(x) > 1$ can not hold, for all $x \in [0,1]$ since equation (17) would vanish identically, implying $\widetilde{S} \equiv 0$. Suppose that $\widetilde{S}(x) > 1$ for $x \in I$, where $I$ is a non-degenerate interval in $[0,1]$, maximal with that property. Then we get $-d\widetilde{S}_{xx} + \widetilde{S}_x = 0$ on $I$ and at least one endpoint $y$ of $I$ must be an interior point of $[0,1]$ satisfying $\widetilde{S}(y) = 1$. Let $y_1, y_2$ be the endpoints of $I$ with $y_1 < y_2$. Integrating the above equation over $I$ leads to

$$d\widetilde{S}'(y_2) - \overline{S}(y_2) - d\widetilde{S}'(y_1) + \widetilde{S}(y_1) = 0. \tag{18}$$

If $y_1 = 0$, then $y_2 < 1$ with $\widetilde{S}(y_2) = 1$ and $\widetilde{S}'(y_2) \le 0$ since for all $y \in I$, $\widetilde{S}(y) > \widetilde{S}(y_2) = 1$. Furthermore, using the boundary conditions, (18) becomes $d\widetilde{S}'(y_2) - 1 = 0$ which leads to a contradiction to $\widetilde{S}'(y_2) \le 0$.
If $0 < y_1 < y_2 < 1$, then $\widetilde{S}(y_1) = \widetilde{S}(y_2) = 1$, $\widetilde{S}'(y_1) \ge 0$ and $\widetilde{S}'(y_2) \le 0$. So, (18) becomes $\widetilde{S}'(y_1) = \widetilde{S}'(y_2) = 0$. Thus, using (16), it yields that $\widetilde{S} \equiv 1$ on $I$ which leads to a contradiction to $\widetilde{S}(x) > 1$, for $x \in I$.
If $0 < y_1 < y_2 = 1$, then $\widetilde{S}(y_2) \ge 1$, $\widetilde{S}'(y_1) \ge 0$ and $\widetilde{S}(y_1) = 1$. So, since $\widetilde{S}'(1) = 0$, (18) becomes $1 - \widetilde{S}(y_2) - d\widetilde{S}'(y_1) = 0$ and implies that $\widetilde{S}'(y_1) = 0$ so $\widetilde{S}(y_2) = 1 = \widetilde{S}(y_1)$. Thus, $\widetilde{S} \equiv 1$ on $I$ which leads to a contradiction to $\widetilde{S}(x) > 1$ for $x \in I$.
Hence, $0 \le \widetilde{S}(x) \le 1$, for all $x \in [0,1]$.

□

## 4. Well-posedness for regular data

Now, we can show the global existence and uniform boundedness solution of system (1)-(3), under the assumption on control of mass ($\mathbf{H_2}$).

### 4.1. *Proof of theorem1*

Let $\varphi : \mathbb{R} \to [0,1]$ be a smooth function such that $\varphi(t) = 0$ for $t \leq 0$, $\varphi(t) = 1$ for $t \geq 1$ and $0 \leq \varphi\prime(t) \leq 2$ for all $t \in \mathbb{R}$. Define the shifted function $\varphi_\tau(\cdot) = \varphi(\cdot - \tau)$. Then the function $\varphi_\tau u$ is a solution of the following system

$$\begin{cases} (\varphi_\tau u)_t - d_1(\varphi_\tau u)_{xx} + (\varphi_\tau u)_x = F_\tau(S,u,v),\ x \in (0,1),\ t \in (\tau, \tau+2), \\ -d_1(\varphi_\tau u)_x(0,t) + (\varphi_\tau u)(0,t) = 0,\ (\varphi_\tau u)_x(1,t) = 0,\ t \in (\tau, \tau+2), \\ (\varphi_\tau u)(x,\tau) = 0, x \in (0,1). \end{cases} \quad (19)$$

where $F_\tau(S,u,v) := \varphi_\tau f(S)\varphi_\tau u - \dfrac{1}{y_u}\varphi_\tau \alpha(u,v)u + \varphi_\tau \beta(u,v)v + \varphi'_\tau u.$
Thanks to the polynomial growth $(\mathbf{H_2})$, we have that

$$F_\tau(S,u,v) \leq C(1+u+v)^r.$$

Using Lemma 1 and Lemma 2, we deduce that for any $1 \leq p < +\infty$, there exists $C_p > 0$, such that

$$\int_\tau^{\tau+2} \int_0^1 (F_\tau(S,u,v))^p dxdt \leq C_p.$$

This implies, using principles comparaison, that

$$\|\varphi_\tau u\|_{L^\infty((0,1)\times(\tau,\tau+2))} \leq C,$$

where $C$ is independent of $\tau \in \mathbb{N}$.
Since $\varphi_\tau \geq 0$ and $\varphi_\tau(t) = 1,\ t \in (\tau+1, \tau+2)$, then $\|u\|_{L^\infty((0,1)\times(\tau,\tau+2))}$ is uniformly bounded.
Similarly, we prove that $\|v\|_{L^\infty((0,1)\times(\tau,\tau+2))}$ is uniformly bounded.

### 4.2. *Proof of theorem 2*

Denote by $\phi := \phi_d$ the principal eigenfunction associated to $\lambda_d$. Then, thanks to $(\mathbf{H_3})$, we claim that, for $c > 0$ large enough we have

$$\alpha(\frac{\phi}{c}, \frac{\phi}{c})(\frac{1}{y_u} - \frac{1}{k}) \leq f(1) - \lambda_d \leq \alpha(c\phi, c\phi)(\frac{1}{y_u} - k),$$

and

$$\alpha(\frac{\phi}{c}, \frac{\phi}{c})k(\frac{1}{y_v} - \frac{1}{k}) \leq g(1) - \lambda_d \leq \alpha(c\phi, c\phi)\frac{1}{k}(\frac{1}{y_v} - k).$$

Let $\underline{w} = (0, \frac{\phi}{c}, \frac{\phi}{c})$. Then, we have

$$\begin{cases} D\underline{w}_{xx} - \underline{w}_x + F(\underline{w}) = G(\underline{w}),\ x \in (0,1), \\ -d\underline{w}_x(0) + \underline{w}(0) = 0,\ \underline{w}_x(1) = 0. \end{cases}$$

where

$$G(\underline{w}) := \begin{pmatrix} f(1)\frac{\phi}{c} + g(1)\frac{\phi}{c} \\ -\lambda_d\frac{\phi}{c} + f(1)\frac{\phi}{c} - \frac{1}{y_u}\alpha(\frac{\phi}{c},\frac{\phi}{c})\frac{\phi}{c} + \beta(\frac{\phi}{c},\frac{\phi}{c})\frac{\phi}{c} \\ -\lambda_d\frac{\phi}{c} + g(1)\frac{\phi}{c} + \alpha(\frac{\phi}{c},\frac{\phi}{c})\frac{\phi}{c} - \frac{1}{y_v}\beta(\frac{\phi}{c},\frac{\phi}{c})\frac{\phi}{c} \end{pmatrix}.$$

thanks to $(\mathbf{H_3})$, by an elementary calculus, we obtain that

$$\begin{aligned} -\lambda_d\frac{\phi}{c} + f(1)\frac{\phi}{c} - \frac{1}{y_u}\alpha(\frac{\phi}{c},\frac{\phi}{c})\frac{\phi}{c} + \beta(\frac{\phi}{c},\frac{\phi}{c})\frac{\phi}{c} &\geq [-\lambda_d + f(1) - \frac{1}{y_u}\alpha(\frac{\phi}{c},\frac{\phi}{c}) + \frac{1}{k}\alpha(\frac{\phi}{c},\frac{\phi}{c})]\frac{\phi}{c} \\ &= [-\lambda_d + f(1) + (\frac{1}{k} - \frac{1}{y_u})\alpha(\frac{\phi}{c},\frac{\phi}{c})]\frac{\phi}{c} \geq 0. \end{aligned}$$

$$\begin{aligned} -\lambda_d\frac{\phi}{c} + g(1)\frac{\phi}{c} - \frac{1}{y_v}\beta(\frac{\phi}{c},\frac{\phi}{c})\frac{\phi}{c} + \alpha(\frac{\phi}{c},\frac{\phi}{c})\frac{\phi}{c} &\geq [-\lambda_d + g(1) - \frac{k}{y_v}\alpha(\frac{\phi}{c},\frac{\phi}{c}) + \alpha(\frac{\phi}{c},\frac{\phi}{c})]\frac{\phi}{c} \\ &= [-\lambda_d + g(1) + (1 - \frac{k}{y_v})\alpha(\frac{\phi}{c},\frac{\phi}{c})]\frac{\phi}{c} \geq 0. \end{aligned}$$

Thus, $\underline{w} = (0, \frac{\phi}{c}, \frac{\phi}{c})$ is a lower solution for the problem (13).
Now, let $\overline{w} = (1, c\phi, c\phi)$. Then, we have

$$\begin{cases} D\overline{w}_{xx} - \overline{w}_x + F(\overline{w}) = G(\overline{w}), \ x \in (0,1), \\ -d\underline{w}_x(0) + \overline{w}(0) \geq 0, \ \overline{w}_x(1) = 0. \end{cases}$$

where

$$G(\overline{w}) := \begin{pmatrix} f(0)c\phi + g(0)c\phi \\ -\lambda_d c\phi + f(0)c\phi - \frac{1}{y_u}\alpha(c\phi,c\phi)c\phi + \beta(c\phi,c\phi)c\phi \\ -\lambda_d c\phi + g(0)c\phi + \alpha(c\phi,c\phi)c\phi - \frac{1}{y_v}\beta(c\phi,c\phi)c\phi \end{pmatrix}.$$

Thanks to $(\mathbf{H_3})$, by an elementary calculus, we obtain that

$$\begin{aligned} -\lambda_d c\phi + f(0)c\phi - \frac{1}{y_u}\alpha(c\phi,c\phi)c\phi + \beta(c\phi,c\phi)c\phi &\leq [-\lambda_d - \frac{1}{y_u}\alpha(c\phi,c\phi) + k\alpha(c\phi,c\phi)]c\phi \\ &= [-\lambda_d + (k - \frac{1}{y_u})\alpha(c\phi,c\phi)]c\phi \leq 0. \end{aligned}$$

$$\begin{aligned} -\lambda_d c\phi + g(0)c\phi - \frac{1}{y_v}\beta(c\phi,c\phi)c\phi + \alpha(c\phi,c\phi)c\phi &\leq [-\lambda_d - \frac{1}{ky_v}\alpha(\frac{\phi}{c},\frac{\phi}{c}) + \alpha(c\phi,c\phi)]c\phi \\ &= [-\lambda_d + (1 - \frac{1}{ky_v})\alpha(c\phi,c\phi)]c\phi \leq 0. \end{aligned}$$

Thus, $\overline{w} = (1, c\phi, c\phi)$ is an upper solution for the problem (13).
Furthermore, $F$ is continuously differentiable function on $[\underline{w}, \overline{w}] \in [0,1]^3$, by normalizing $\phi$, then for $\lambda$ large enough, the mapping $w_i \longmapsto \lambda w_i + F_i(\cdot, w_i, \cdot)$ is nondecreasing on $[\underline{w}_i, \overline{w}_i], i \in \{1,2,3\}$.

Then by virtue of Lemma 4, there exist two extremal solutions in $[\underline{w}, \overline{w}]$ of the problem (13). That is there exists a non-trivial steady-state $(S, u, v)$ of (11) satisfying for $x \in [0, 1]$,

$$\frac{1}{c}\phi(x) \leq u(x) \leq c\phi(x) \text{ and } \frac{1}{c}\phi(x) \leq v(x) \leq c\phi(x),$$

$$\text{that is } u(x) \approx v(x) \approx \phi(x).$$

Which ends the proof.

### 4.3. *Proof of theorem 3*

The linearization of (1) at the washout steady-state is given by:

$$\begin{cases} S_t = dS_{xx} - S_x - f(1)u - g(1)v \\ u_t = du_{xx} - u_x + f(1)u \\ v_t = dv_{xx} - v_x + g(1)v. \end{cases} \tag{20}$$

Denote by $\overline{S} = 1 - S$. The relevant eigenvalue problem for the stability of the washout steady-state writes:

$$(E_\lambda) \begin{cases} \lambda\overline{S} = d_0\overline{S}'' - \overline{S}' - f(1)u - g(1)v \\ \lambda u = d_1u'' - u' + f(1)u \\ \lambda v = d_2v'' - v' + g(1)v, \end{cases} \tag{21}$$

with the homogeneous boundary conditions:

$$\begin{cases} -d\overline{S}'(0) + \overline{S}(0) = 0, \ \overline{S}'(1) = 0 \\ -du'(0) + u(0) = 0, \ u'(1) = 0 \\ -dv'(0) + v(0) = 0, \ v'(1) = 0. \end{cases} \tag{22}$$

The eigenvalues $\lambda$ of $(E_\lambda)$ correspond to the eigenvalues $\lambda_n$ of $(P_\lambda)$ or $\lambda_n + f(1)$ or $\lambda_n + g(1)$
Then, $\lambda$ verifies the following inequalities:

$$\lambda \leq -\lambda_d \text{ or } \lambda \leq -\lambda_d + g(1) \text{ or } \lambda \leq -\lambda_d + f(1).$$

So, if $\lambda_d > g(1)$ and $\lambda_d > f(1)$, then $\lambda < 0$ and the washout steady-state is stable, which ends the proof.

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